Mathematical Methods

Unit 3 – Problem-solving and modelling task

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1 Introduction

This Problem-solving and modelling task is to design a profile of a roller coaster and calculate its excitement factor. To come up with the design, a minimum of 3 types of functions including polynomial, trigonometric, exponential and logarithmic are used to generate the model. The excitement factor depends on the angle of steepest descent in each drop and the total vertical distance of the top. Both technological and mathematical procedures are used to design the profile, including simultaneous equation solving, calculus and trigonometry, and technology, for example Excel, graphic calculator and Desmos. The report also includes evaluation and recommendations to further discuss the strength and limitation of the model.

2 Considerations

2.1 Observations

1. It was observed that the height of the roller coaster has to be within [0, 90] and the horizontal distance is within [0, 250].

2. It was observed that the ride is smooth and the slope is defined everywhere on its domain.

3. It was observed that the angle of the steepest descent of the roller coaster does not exceed 80 degrees.

4. It was observed that the ride consists of at least 3 different types of the functions from polynomial, trigonometric, exponential and logarithmic.

2.2 Assumptions

1. Since the height of the roller coaster is within [0, 90] and the horizontal distance is within [0, 250], it is assumed that the domain of the roller coaster is [0, 230] and range is [0, 88.5]

2. Base on assumption 1, it is assumed the roller coaster starts at (0,15) and ends at (230,0).

3. According to basic physics principles, the first hill of the roller coaster is assumed to be the highest hill of the ride (Liddle, 2013).

3 Mathematical concepts and procedures

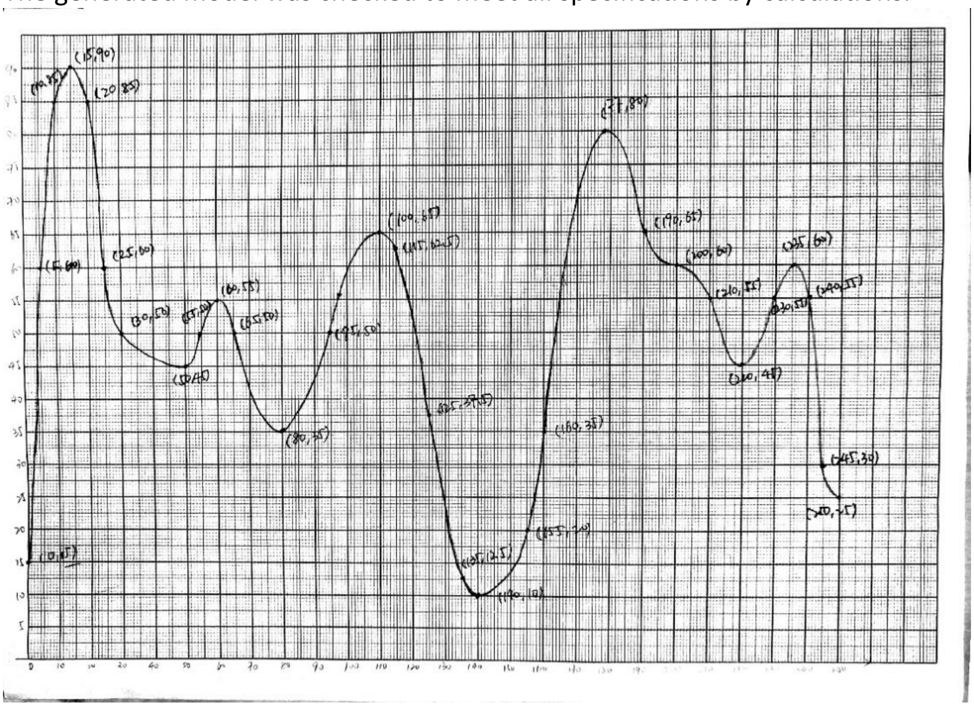
To come up with the functional model of the roller coaster, the initial method was to draw the design of the roller coaster on a sketching paper and input the coordinates of key points into Excel for the equations to be generated. However, since the major process was done manually, the design was inaccurate and did not satisfy the specifications. The new method was to calculate the derivative of the end point of the previous function first and then by solving simultaneous equations, the equation of the next function was generated. The refinement of the model was further investigated based on the smoothness, the angles of the descent and the excitement factor. By this way, the functional model has been developed to satisfy all the specifications and at the same time achieved a great value of excitement factor.

4 Determining the solution and refinement of the model

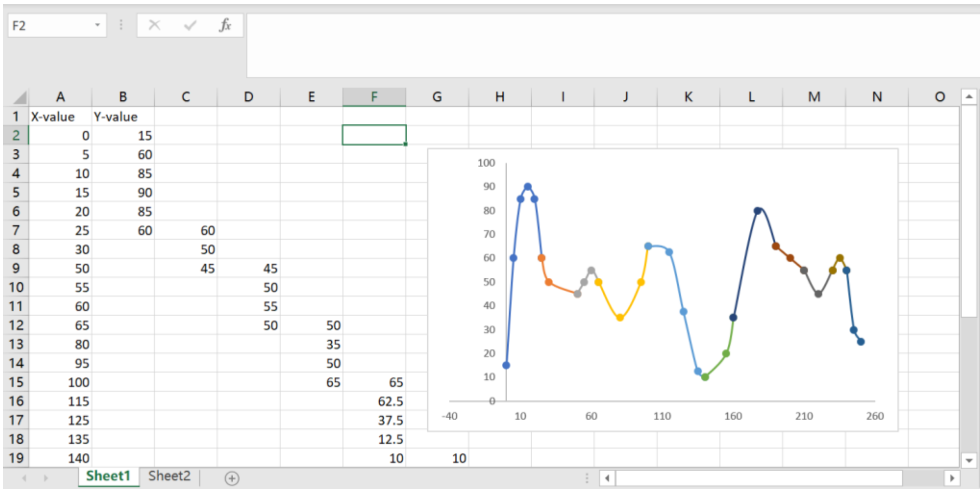
4.1 Generate the profile model

Initial method:

1. The profile of the roller coaster was designed and sketched on a graphing paper.
2. The key points of each function on the profile were found and input into Excel.
3. A scetter graph was generated and trend lines were used to find the equation of each function.
4. The generated model was checked to meet all specifications by calculations.



(Figure 1: A photograph of the design of the roller coaster on a skteching paper with all the selected key points marked.)



	A	B	C	D	E	F	G
1	X-value	Y-value					
2	0	15					
3	5	60					
4	10	85					
5	15	90					
6	20	85					
7	25	60	60				
8	30		50				
9	50		45	45			
10	55			50			
11	60			55			
12	65			50	50		
13	80				35		
14	95				50		
15	100				65	65	
16	115					62.5	
17	125					37.5	
18	135					12.5	
19	140					10	10

(Figure 2: Graph showing the initial model of the roller coaster.)

New method:

1. One point (does not coincide with the starting point) was found within the domain and range of the functional model and it was connected with the starting point by adding an exponential trend line.
2. The equation of Function 1 was $y=15e^{0.0693x}$ given by Excel. Function 1 was extended to when x=20, so substitutes x=20 into the equation:

$$f1(x)=15e^{0.0693*20}=60$$

Function 1 ends at (20,60).
3. As Function 1 is an exponential function, the derivative of its end point can be found using the chain rule. If $y = e^{f(X)}$, let u = f(x), then $y = e^n$.

$$\frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = f'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$f1'(x) = e^{0.0693x} \times 0.0693$$

Substitute the x-value of the end point, x = 20.

$$f1'(x) = e^{0.0693\times20} \times 0.0693 = 2.07869$$

Therefore, the derivative of Function 1 at its end point is 2.07869.
4. The next function should continue climbing up and then drop down. So a quadratic was chosen to be Function 2. Let the next function be $f2(x) = ax^2+bx+c$. Since Function 2 is a polynomial function, Differentiation Rule 1, 2 and 3 can be used to determine the derivative at its end point.

$$Rule\ 1: If\ f(x) = x^n, then\ f'(x) = nx^{n-1}$$

$$Rule\ 2: If\ f(x) = ax^n, then\ f'(x) = nax^{n-1}$$

$$Rule\ 3: If\ f(x) = c, then\ f'(x) = 0\ (where\ c\ is\ constant)$$

$$f2'^{(x)} = 2ax + b$$
5. The coordinate of the end point of the previous function was substituted into $y= ax^2+bx+c$:

$$60 = 20^2a + 20b + c = 400a + 20b + c \quad ①$$

The derivative of the end point as well as the x value was substituted into f2'(x) = 2ax+b:

$$2.07869 = 2 \times 20a + b = 40a + b \quad ②$$

Rearrange equation ①, $60 - c = 400a + 20b$ ③

$$② \times 10: 20.7869 = 400a + 10b \quad ④$$

$$③ - ④: 60 - c - 20.7869 = 10b$$

$$b = \frac{60 - c - 20.7869}{10}$$
6. Since there were 3 variables, but only 2 simultaneous equations at start, c needed to be guessed a constant value. The shape of the parabola has a maximum, so a has to be a negative number. According to equation ②, when a negative, b has to be greater than 2.07869. To make b greater than 2.07869:

$$\frac{60 - c - 20.7869}{10} \geq 2.07869$$

$$c \leq 18.4262$$

Therefore, c can be any value smaller or equal to 18.4262. Let c be 1.

$$b = \frac{60 - c - 20.7869}{10} = \frac{60 - 1 - 20.7869}{10} = 3.82131$$

Substitute b in ②, $a = \frac{2.07869-3.82131}{400} = -0.043566$

7. The equation of Function 2 was determined to be $f2(x) = y=-0.043566x^2+3.82131x+1$. The turning point was found by applying the second derivative test:

$$let\ f2'(x) = 0$$

$$0 = 2 \times (-0.043566)x + b$$

$$x = 43.9,$$

$$y(43.9) = -0.043566 \times 43.9^2 + 3.82131 \times 43.9 + 1 = 84.8$$

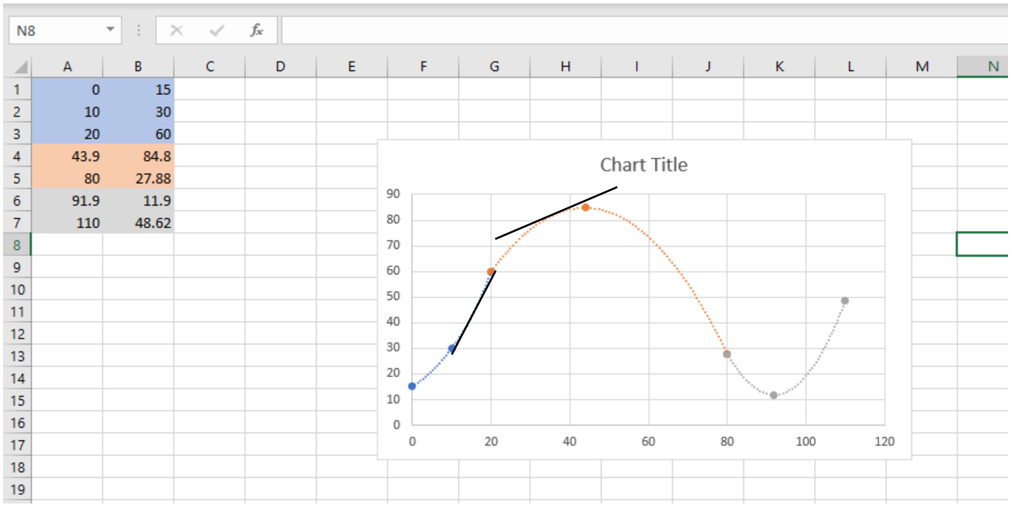
$$f2''(x) = 2 \times (-0.043566) = -0.087132 < 0$$

Therefore, the turning point of Function 2 is (43.9, 84.8) and as f2''(x) < 0, (43.9,84.8) is a local maximum. The end point of Function 2 is when x = 80. Its y-value was determined by substituting x = 80 into the equation.

$$y = -0.043566 \times 80^2 + 3.82131 \times 80 + 1 = 27.88$$

Therefore, Function 2 ends at (80,27.88)

8. The next function should continue going down and then go up, so it was chosen to be another quadratic. The steps used to determine Function 2 were repeated in determining Function 3.
9. f3(x) = $0.1127x^2$-20.72x+964.25. The turning point is (91.9, 11.9) and as f3''(x) < 0, it is a local minimum. The end point is (110,48.62) and its derivative is 4.074.



(Figure 3: Graph showing the derivatives of the first descent.)

10. Angle of steepest descent is at the connecting point between Function 2 and 3: When x =79.9, y = 28.1969.

$$\tan\theta = \frac{|27.88 - 28.1969|}{80 - 79.9}$$

$$\theta = \tan^{-1}\frac{|27.88 - 28.1969|}{80 - 79.9} = 72.49°$$

Excitement Factor: $72.49 \times \frac{\pi}{180} = \frac{72.49}{180}\pi$

$$\frac{72.49}{180}\pi \times (84.8 - 11.9) = 92.2323$$

11. The next function needs to continue climbing up and then goes down. So it was chosen to be a parabola. The steps used to determine Function 2 were repeated again to determine Function 4.
12. f4(x)= $-0.03003x^2$+72.205x-4260.4. The turning point is (120,80) and as f4''(x) < 0, it is a local maximum. The end point is (126,70) and its derivative is -3.47. The next function is exponential as it continues going down. Let f5(x) = ke^{ax}.

$$f5'(x) = kae^{ax}$$

Substitute the end point of Function 4: $70 = ke^{126a}$

Substitute the derivative and x-value: $-3.47 = kae^{126a}$

$$a = -0.033$$

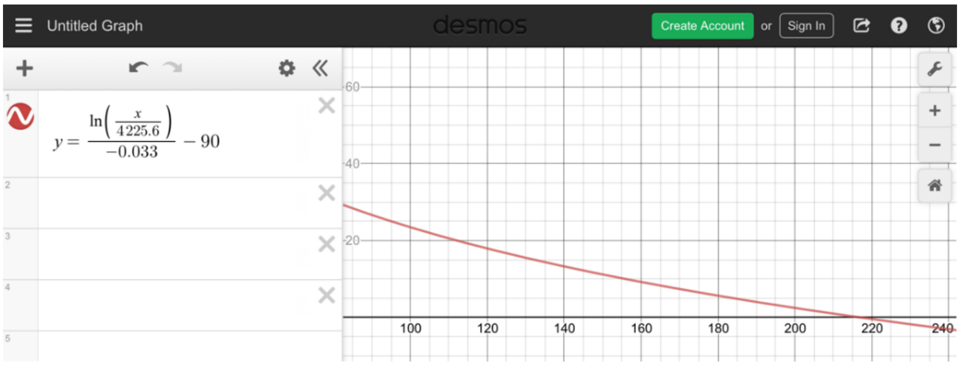
Substitute a into one of the simultaneous equations: $k = 4225.6$.

13. f5(x) = $4225.6e^{-0.033x}$ and the derivative at its end point is -0.511. As the value of its derivative is small, a logarithmic function would fit as the last part of the roller coaster. Let f6(x) = aln(x)+b. To make sure Function 5 and 6 connected smoothly, Function 6 can be determined by converting Function 5 into a logarithmic function.

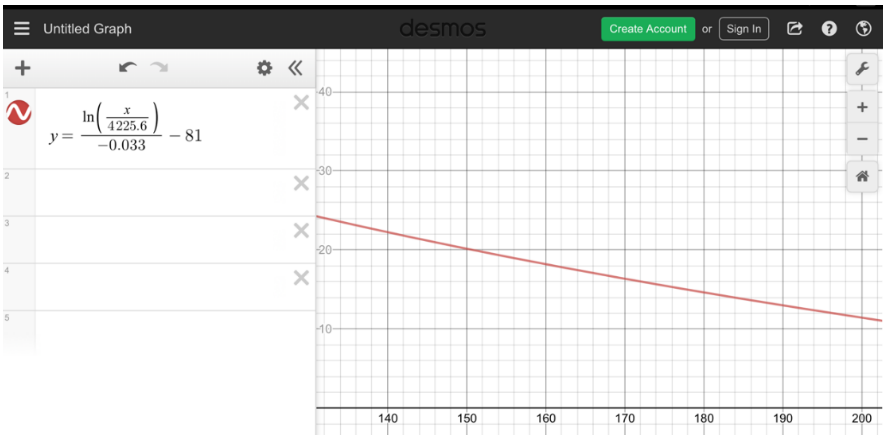
$$y = 4225.6e^{-0.033x}$$
$$\frac{y}{4225.6} = e^{-0.033x}$$
$$-0.033x = \ln\left(\frac{y}{4225.6}\right)$$
$$x = \frac{\ln\left(\frac{y}{4225.6}\right)}{-0.033}$$
$$let\ f6(x) = \frac{\ln\left(\frac{x}{4225.6}\right)}{-0.033} - a$$

As a is a constant, it does not affect the derivative of the function. Desmos and graphic calculator were used to help to determine the value of a. (evidence shown in Figure 4-8)

(Figure 4: Screenshot of Function 6 on Desmos)

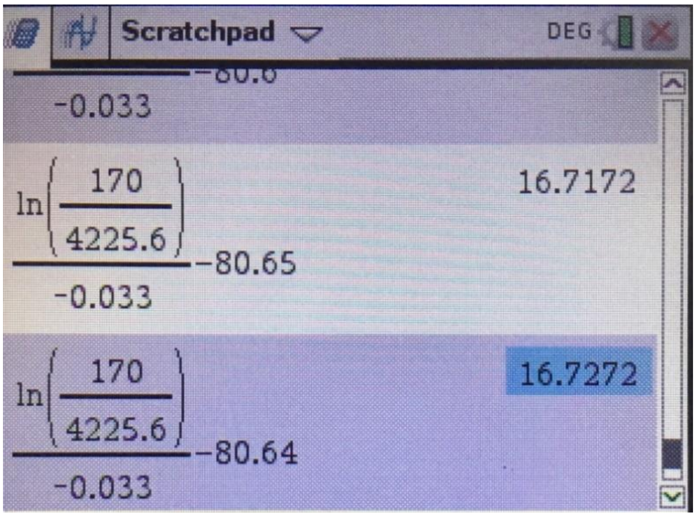


(Figure 5: Screenshot of Function 6 on Desmos)



(Figure 6: Screenshot of Function 6 on Desmos)

(Figure 7: Screenshot of Function 6 on Desmos)

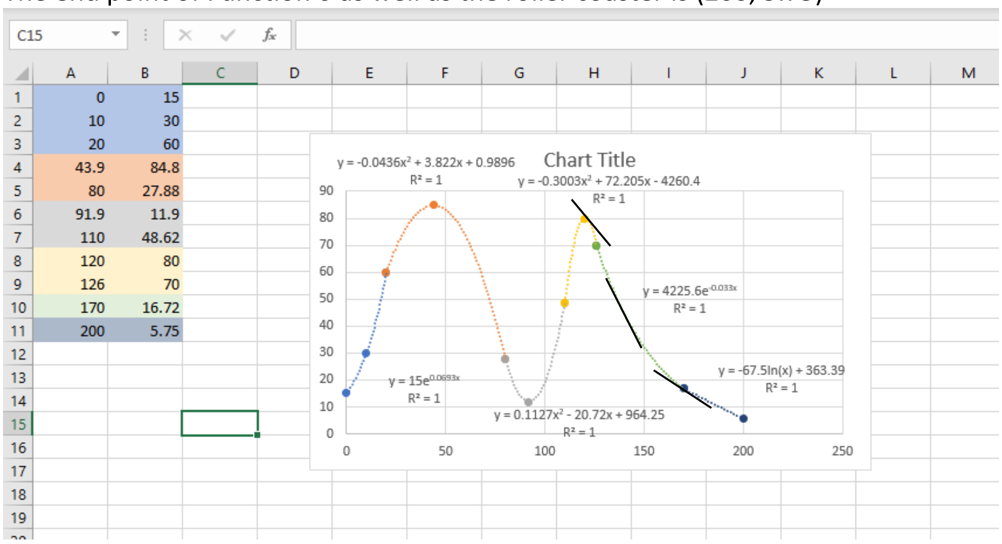


(Figure 8: Screenshot of calculations on graphic calculator when determining Function 6)

f6(x) $= \frac{\ln\left(\frac{x}{4225.6}\right)}{-0.033} - 80.65$. The roller coaster was assumed to finish at x=200. So the ending point was found by substituting its x-value:

$$y = \frac{\ln\left(\frac{200}{4225.6}\right)}{-0.033} = 5.75$$

The end point of Function 6 as well as the roller coaster is (200, 5.75)



	A	B
1	0	15
2	10	30
3	20	60
4	43.9	84.8
5	80	27.88
6	91.9	11.9
7	110	48.62
8	120	80
9	126	70
10	170	16.72
11	200	5.75

(Figure 9: Graph showing the initial model generated by the new method and the derivatives of the second descent.)

14. The angle of steepest descent is at the connecting point between function 4 and 5. It was calculated to be 64.67° and the excitement factor was 83.8063.
15. Total Excitement factor$= 92.2323 + 83.8063 = 176.039$

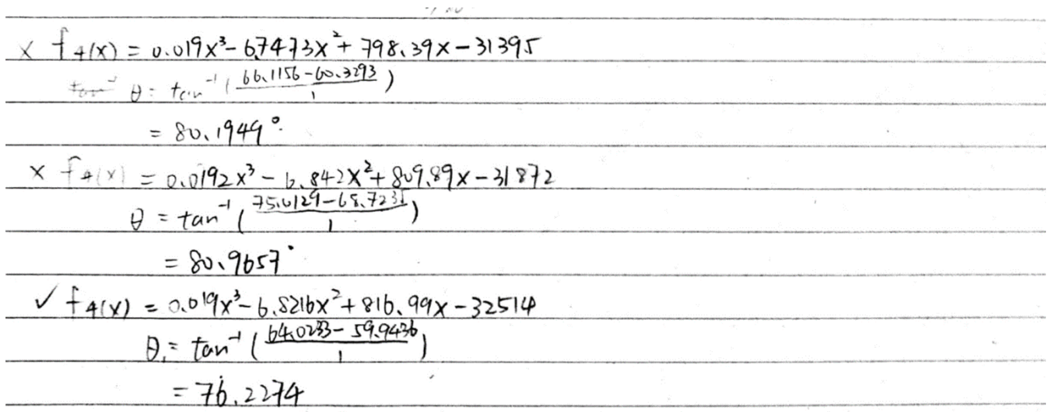
4.2 Development of the profile model

To improve the model, a variety of functions should be added to the design. The first hill as the highest point did not reach 90. To make the roller coaster more exciting, a cubic function was added after Function 3 so that it can continue climbing up after the second turning point. It allows the third hill to exceed 84.8 as long as the y-coordinate of its first turning point is between $90 - (84.8 - 15 + 11.9) = 8.3$ and $84.8 - 15 + 11.9 = 81.7$. Let f4(x) = ax^3+bx^2+cx+d.

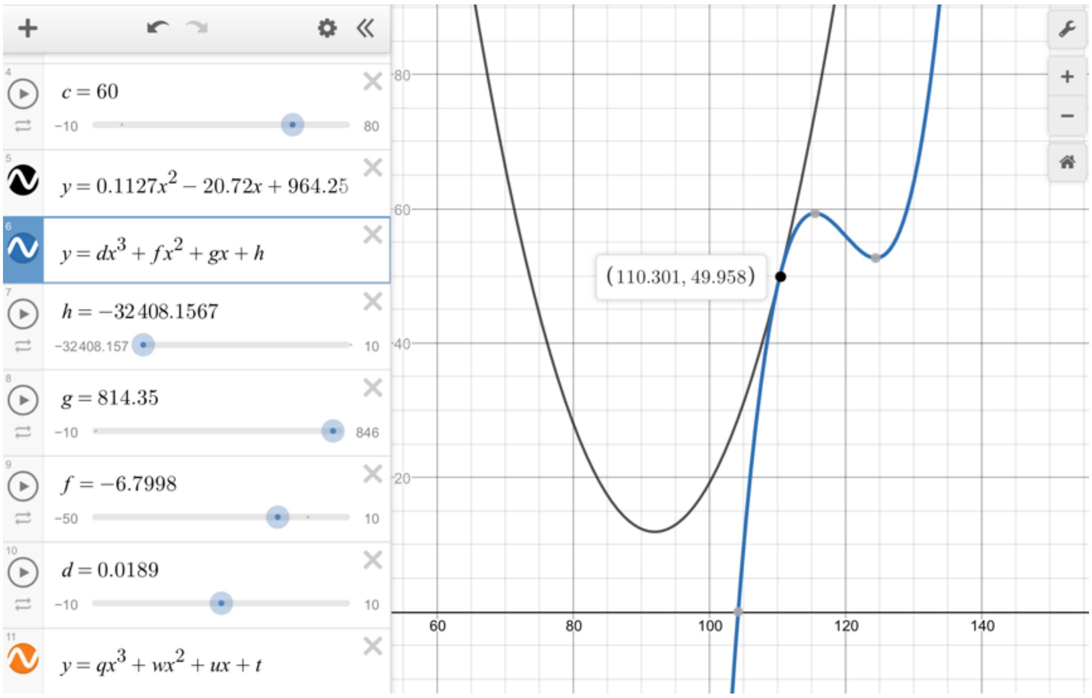
The equation of the derivative of Function 4 is:

$$f4'(x) = 3ax^2 + 2bx + c$$

Since there are 4 variables, but only 2 simultaneous equations, Desmos was used to help to determine Function 4 as it can demonstrate how each constant transfer the graph. As the shape of Function 4 is climbing up, going down and climbing up, a is greater than 0. b and c control the steepness of the slide and d transfers the function up or down. (part of the work shown below in figure 10) After adjusting each factor as well as well as checking their derivatives, Function 4 was determined to be f4(x) = $0.0189x^3-6.7998x^2+814.35x-32408.1567$. (As Figure 11 shown below)



(Figure 10: Photograph of hand working of adjusting the equation to meet the specification)



(Figure 11: Screenshot of Function 3 and 4 on Desmos.)

The end point of Function 4 is (130, 64.02).

The derivative of the end point:

$$f4'(x) = 3 \times 0.0189 \times 130^2 + 2 \times (-6.7998) \times 130 + 814.35 = 6.674$$

The two turning point of Function 4 was (115.477,59.335) and (124.375,52.678). As $f''4(x) = 2 \times 3 \times 0.0189 \times 115.477 + 2 \times (-6.7998) = -0.504508 < 0$, so (155.477,59.335) is a local maximum. As $f''4(x) = 2 \times 3 \times 0.0189 \times 124.375 + 2 \times (-6.7998) = 0.504525 > 0$, so (124.375,52.678) is a local minimum.

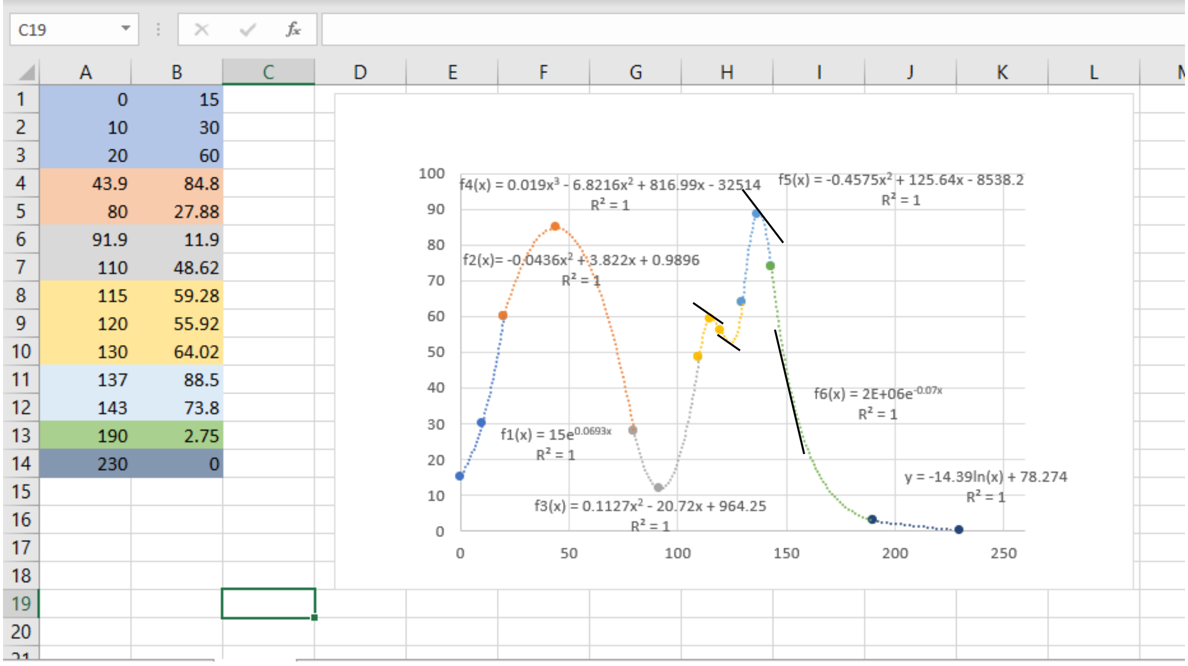
The angle of steepest descent is 76.23° and the excitement factor is 8.8569.

As the end part of Function 4 goes up, but Function 5 goes down, a parabola was needed to connect them. The steps used to determine Function 2 were repeated to determine the parabola as the new Function 5. f5(x) = $-0.4575x^2+125.64x-8538.2$. The turning point of Function 5 is (137,88.5). As f5''(x) < 0, it is a local maximum. The coordinate of its end point is (143, 73.8) and its derivative is -5.16669.

As the derivative has changed, Function 6 and 7 were recalculated to make sure the roller coaster is smooth and its slope is defined everywhere on its domain. (work see appendix)

The angle of steepest drop is at the connecting point between Function 5 and 6, which is 78.95°. The excitement factor is 121.947.

Therefore, the total excitement factor after refinement is $= 92.2323 + 8.8569 + 121.947 = 223.037$



(Figure 12: Graph showing the developed profile of the roller coaster.)

5 Summary of results

Function	Equation	Domain	Range	Type of Function
Function 1	$y=15e^{0.0693x}$	[0,20]	[15,60]	exponential
function 2	$y=-0.0436x^2+3.822x+0.9896$	[20,80]	[27.88,84.8]	polynomial(quadratic)
Function 3	$y=0.1127x^2-20.72x+964.25$	[80,110]	[11.9,48.62]	polynomial(quadratic)
Function 4	$y=0.019x^3-6.8216x^2+816.99x-32514$	[110,130]	[48.62,64.02]	polynomial(cubic)
Function 5	$y=-0.4575x^2+125.64x-8538.2$	[130,143]	[64.02,88.5]	polynomial(quadratic)
Function 6	$y=2E+06e^{-0.07x}$	[143,190]	[2.75,88.5]	exponential
Function 7	$y=14.39\ln(x)+78.274$	[190,230]	[0,2.75]	logarithmic

6 Evaluation

6.1 Overview

During the process of designing the profile of a roller coaster, the functional model was developed to meet all the specifications after all the modifications and refinements. However, since there still are many limitations that could not be avoided.

6.2 Strengths and limitations

This design was based on mathematical concept and had a lot of work applied to various of technology, which means the result was reasonable and relatively accurate. Excel was used in the majority of the process; Desmos was used to help to determine some specific functions as it can show how each constant in the function affects the graph and graphic calculator was used to help with the analysis and development. As a result, the developed profile of the roller coaster meets all the specifications and has a high value of excitement factor. The initial research on basic physical principles of roller coasters also helped to make the design practical. On the other hand, there are still limitations in the modelling process. Firstly, the specifications like types of functions, the maximum angle of steepest descent and the maximum of height and horizontal distance limited the excitement factor of the roller coaster. Secondly, technologies are not perfect. Using three different technology means the result has more sources of error. Thirdly, since the functional model was mainly based on mathematical concepts and the physical and mechanical knowledge was limited, the design may not work in real life.

7 Conclusion

The developed profile of the roller coaster included polynomial, exponential and logarithmic functions and achieved an excitement factor of 223.037. It was generated after a complex and long procedure of determining, development and evaluation. The derivatives at connecting points and the angles of steepest descents were repeatedly checked to make sure they meet the specifications. The application of technology helped in most procedures and the evaluation helped to conclude the report and further investigate the solution achieved.

7.1 Recommendations

1. More types of functions can be used in designing the roller coaster.
2. Applying to higher level technology which will improve the profile itself as well as its accuracy.
3. Referencing existing good examples of roller coasters to help to develop the design.

8 Appendixes

$f_2(x) = ax^2 + bx + 1$

$f_2'(x) = 2ax + b$

$2.07869 = 40a + b$

$60 = 400a + 20b + 1$

$59 = 400a + 20b$

$20.7869 = 400a + 10b$

$b = 3.82131$

$a = -0.043566$

$\therefore f_2(x) = -0.043566x^2 + 3.82131x + 1$

$f_2'(x) = -0.0872x + 3.822 = -0.0872 \times 80 + 3.822 = -3.154$

$f_5(x) = f'_4(x) = 6.674$

$f_5(x) = ax^2 + bx - 8500$ $(137, 8815)$

$f'_5(x) = 2ax + b$

$6.674 = 260a + b$

$14.02 = 130^2 a + 130b - 8500$

$8514.02 = 16900a + 130b$

$433.81 = 16900a + 65b$

$b = 125.08$

$a = -0.455408$

← negative → $b \geq 6.674$ (a ↑ larger, ↑ larger)

$c \leq -165 \times 6.674 + 433.81 - 14.02$ ↓ smaller

$\therefore -0.455408x^2 + 125.08x - 8500$

when $x = 143$ when $x = 142.9$

$y = 73.8018$ $y = 74.3139$

$\theta = \tan^{-1}\left(\frac{73.8018 - 74.3139}{143 - 142.9}\right) = 78.95^\circ$

$73.8 = k e^{143a}$

$a = -0.070009$

$k = 1644004.73$

$\therefore f_6(x) = 1644004.73 e^{-0.070009x}$

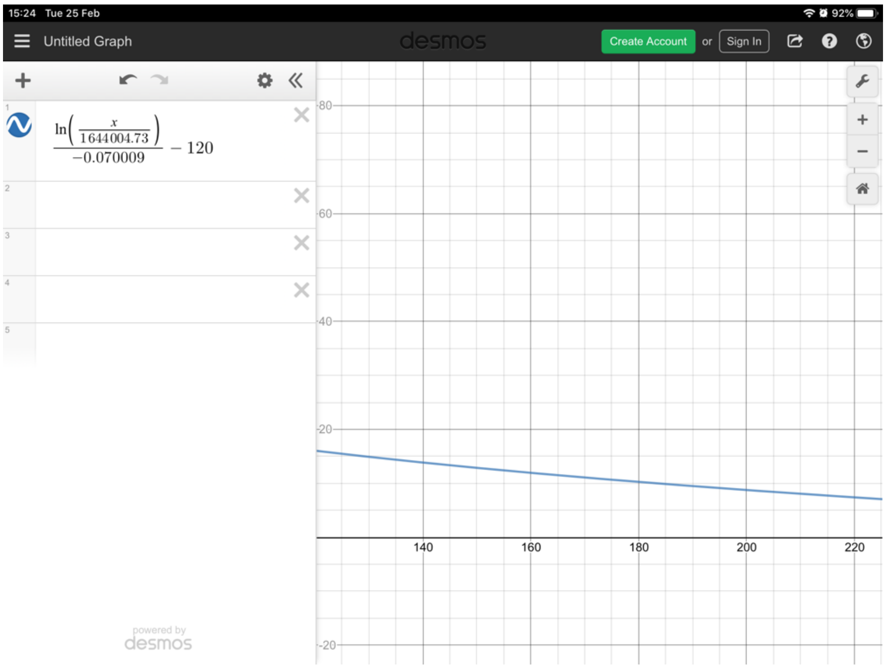
steepness max at the beginning smoothness problem

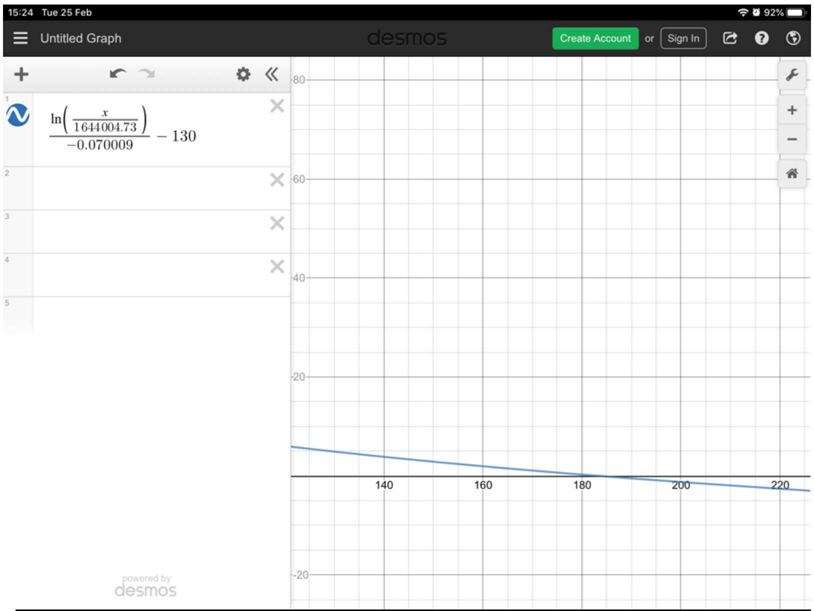
$f'_6(x) = -115095 e^{-0.070009 \times 150} = -3.10504$

$\frac{x}{1644004.73} = e^{-0.070009y}$

$-0.070009y = \ln\left(\frac{x}{1644004.73}\right)$

$y = \ln\left(\frac{x}{1644004.73}\right) \div (-0.070009)$

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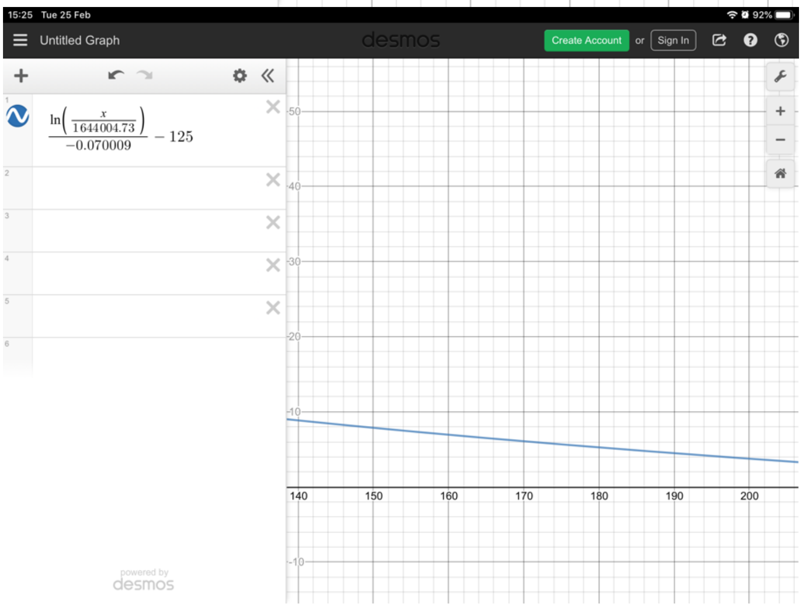
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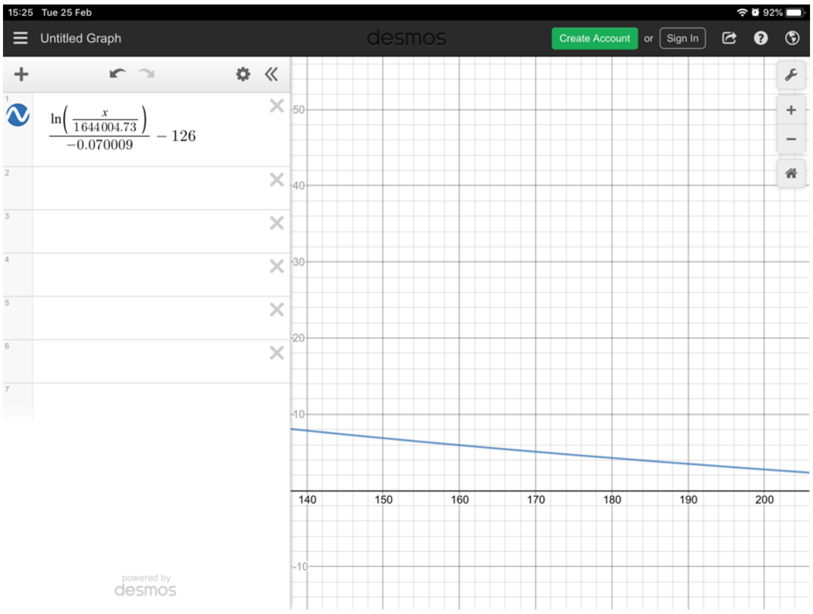
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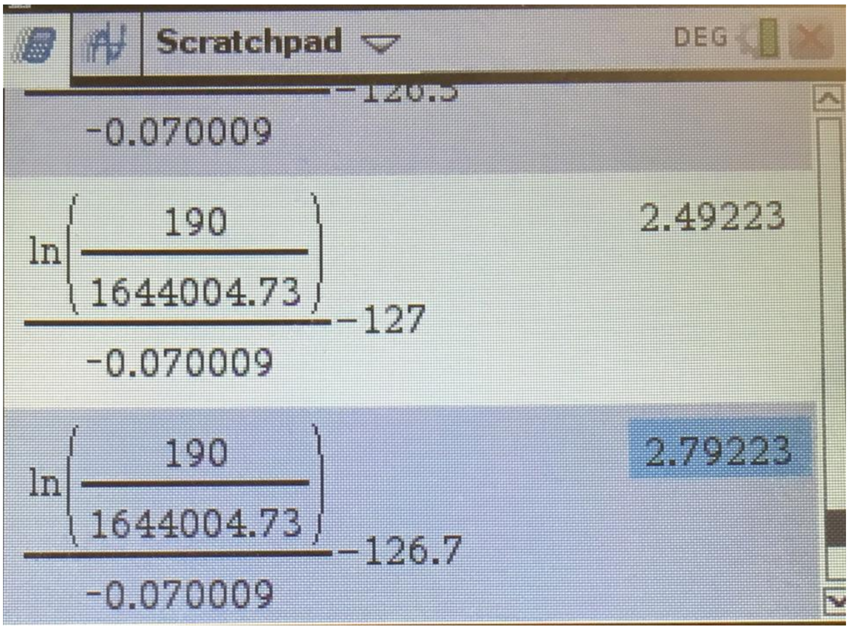
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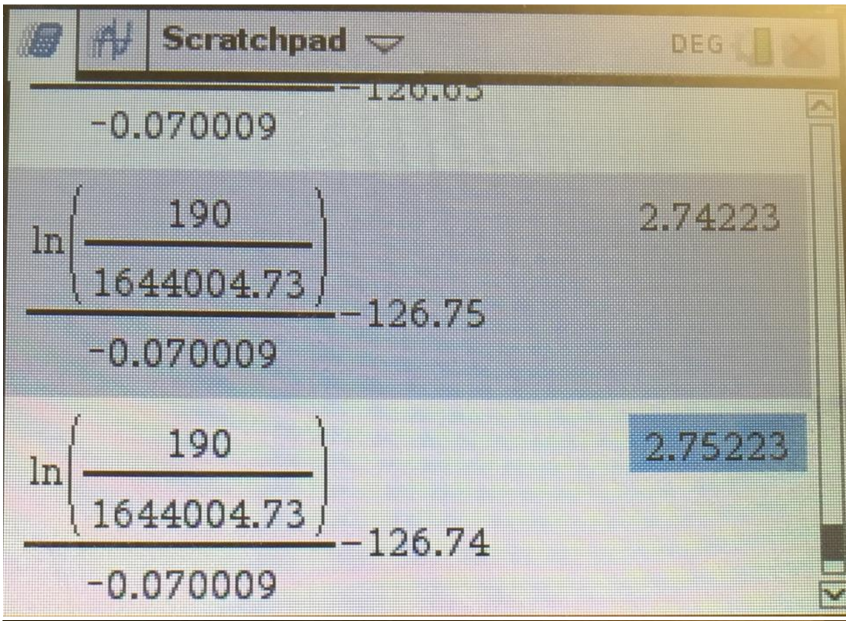
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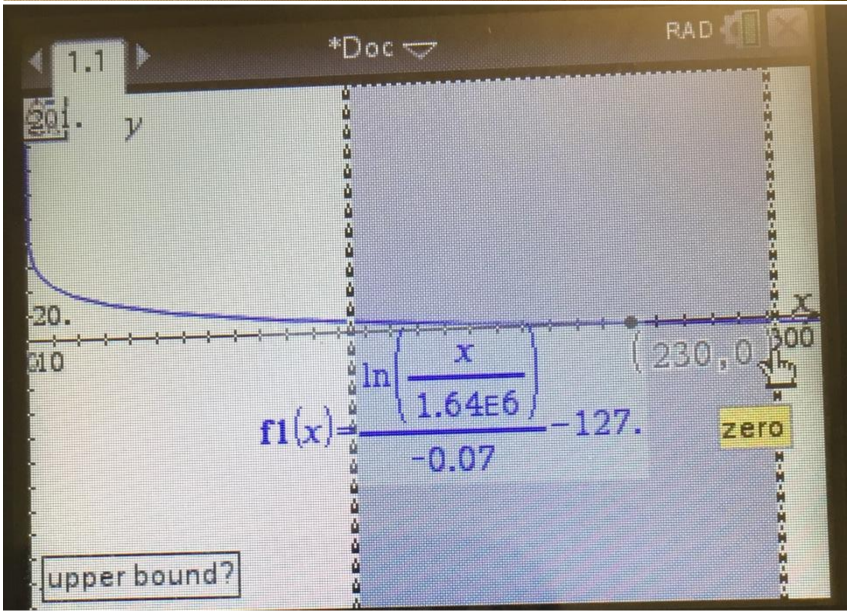
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ln(x/1.64E6) - 127.
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9 Reference list

1. Liddle S. (2013). Lesson: Physics of Roller Coasters. Available at: *https://www.teachengineering.org/lessons/view/duk_rollercoaster_music_less*.